Lecture 18
Fourier Analysis, Low Pass Filters, Decibels

Chapter 6
Frequency Response, Bode Plots, and Resonance

1. State the fundamental concepts of Fourier analysis.

2. Determine the output of a filter for a given input consisting of sinusoidal components using the filter’s transfer function.

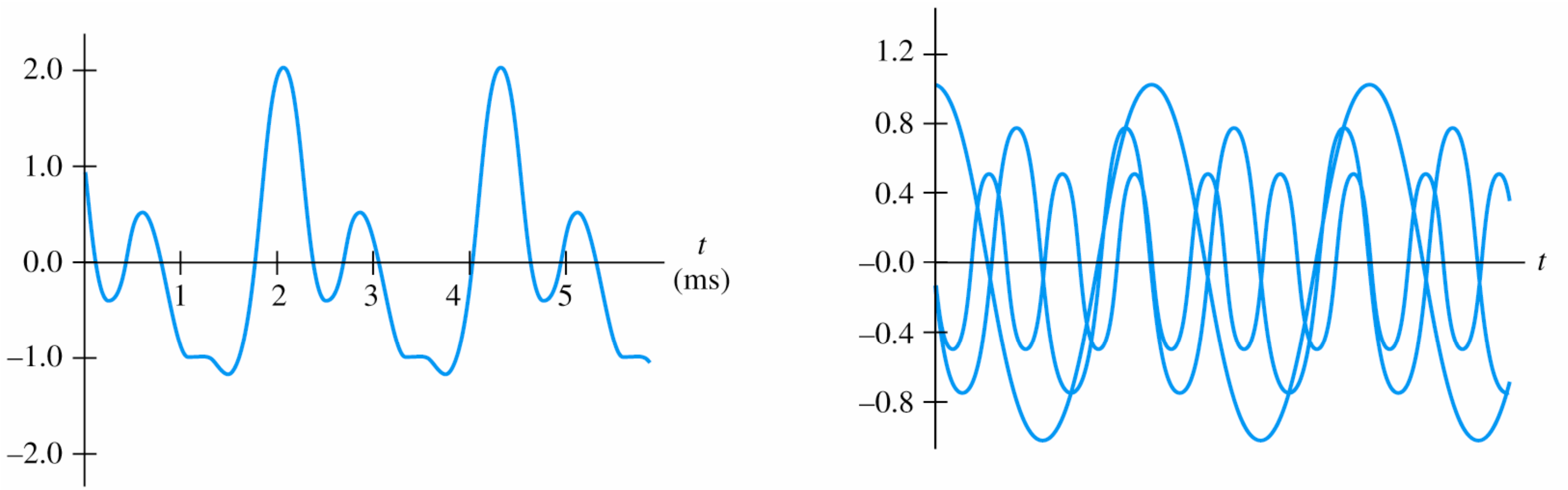
3. Use circuit analysis to determine the transfer functions of simple circuits.

4. Draw first-order lowpass or highpass filter circuits and sketch their transfer functions.

5. Understand decibels, logarithmic frequency scales, and Bode plots.

6. Draw the Bode plots for transfer functions of first-order filters.

Fourier Analysis



(a) Music waveform

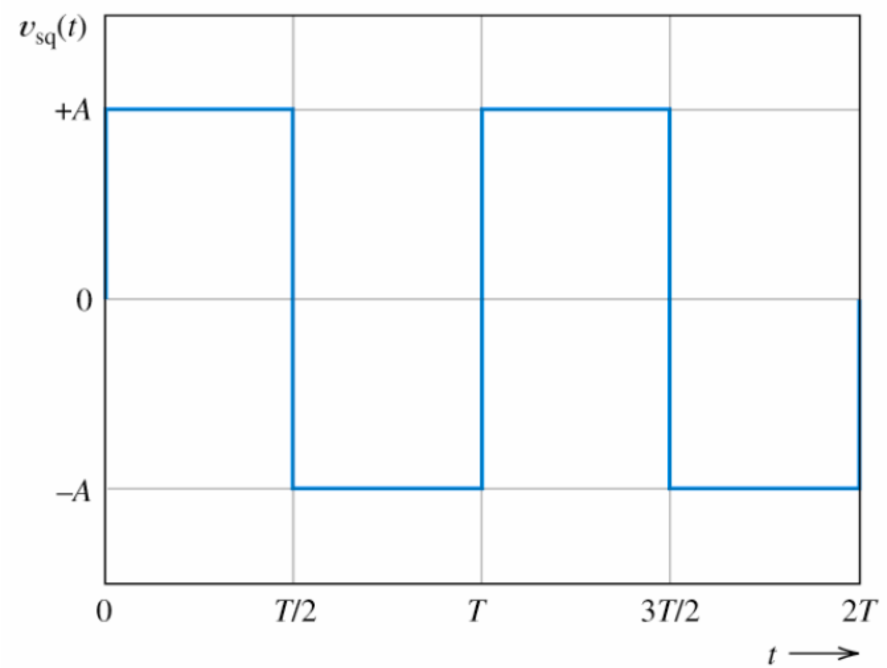
(b) Sinusoidal components

Figure 6.1 The short segment of a music waveform shown in (a) is the sum of the sinusoidal components shown in (b).

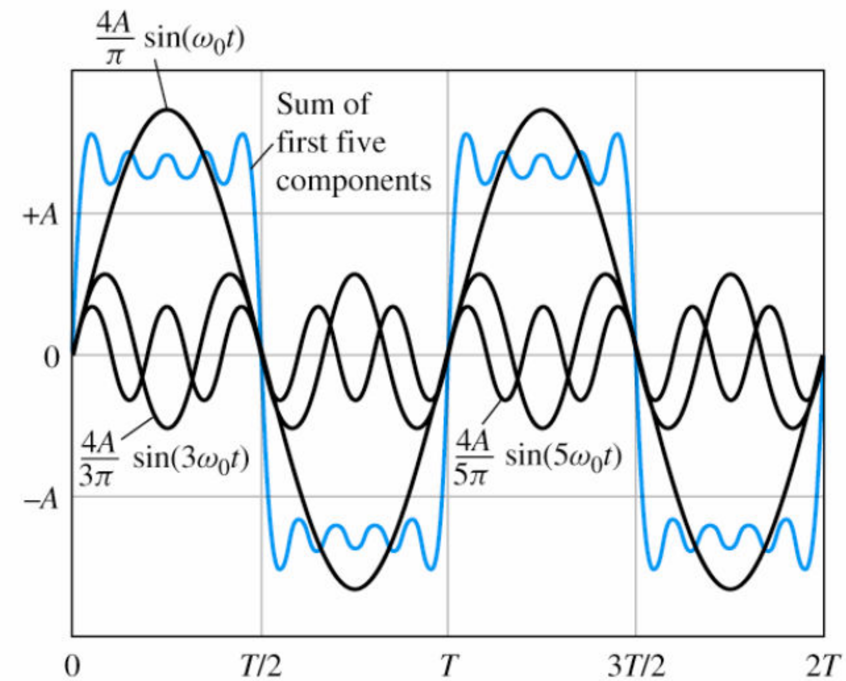
Fourier Analysis

All real-world signals are sums of sinusoidal components having various frequencies, amplitudes, and phases.

Fourier Analysis



(a) Periodic square wave



(b) Several of the sinusoidal components and the sum of the first five components

$$v_{sq}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t) + \frac{4A}{5\pi}\sin(5\omega_0 t) + \ldots.$$

23.7 $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

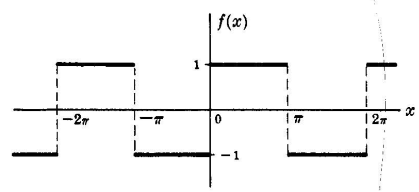


Fig. 23-1

23.8 $f(x) = |x| = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

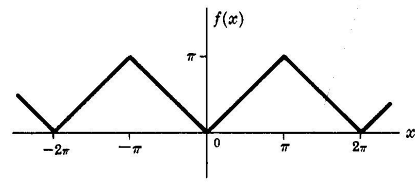


Fig. 23-2

23.9 $f(x) = x, \ -\pi < x < \pi$

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

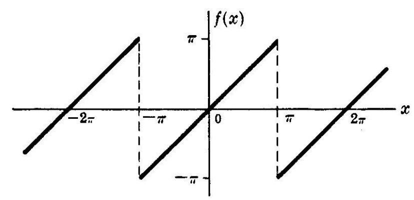


Fig. 23-3

23.10 $f(x) = x, \ 0 < x < 2\pi$

$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

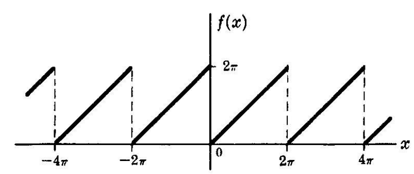


Fig. 23-4

23.11 $f(x) = |\sin x|, \ -\pi < x < \pi$

$$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \cdots\right)$$

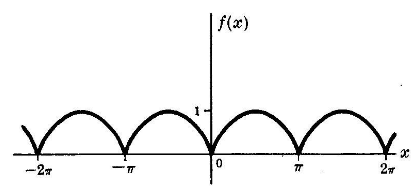


Fig. 23-5

Table 6.1. Frequency Ranges of Selected Signals

Electrocardiogram	0.05 to 100 Hz
Audible sounds	20 Hz to 15 kHz
AM radio broadcasting	540 to 1600 kHz
Video signals (U.S. standards)	Dc to 4.2 MHz
Channel 6 television	82 to 88 MHz
FM radio broadcasting	88 to 108 MHz
Cellular radio	824 to 891.5 MHz
Satellite television downlinks (C-band)	3.7 to 4.2 GHz
Digital satellite television	12.2 to 12.7 GHz

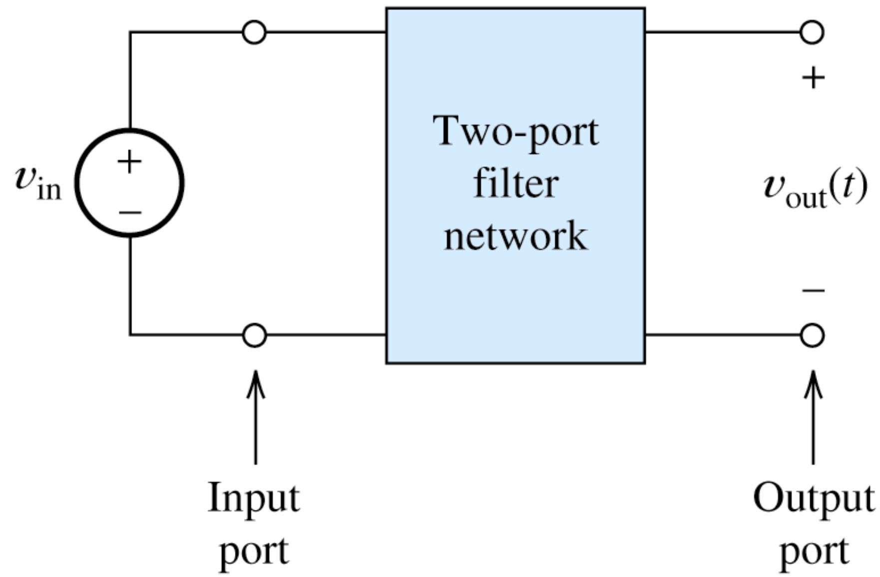
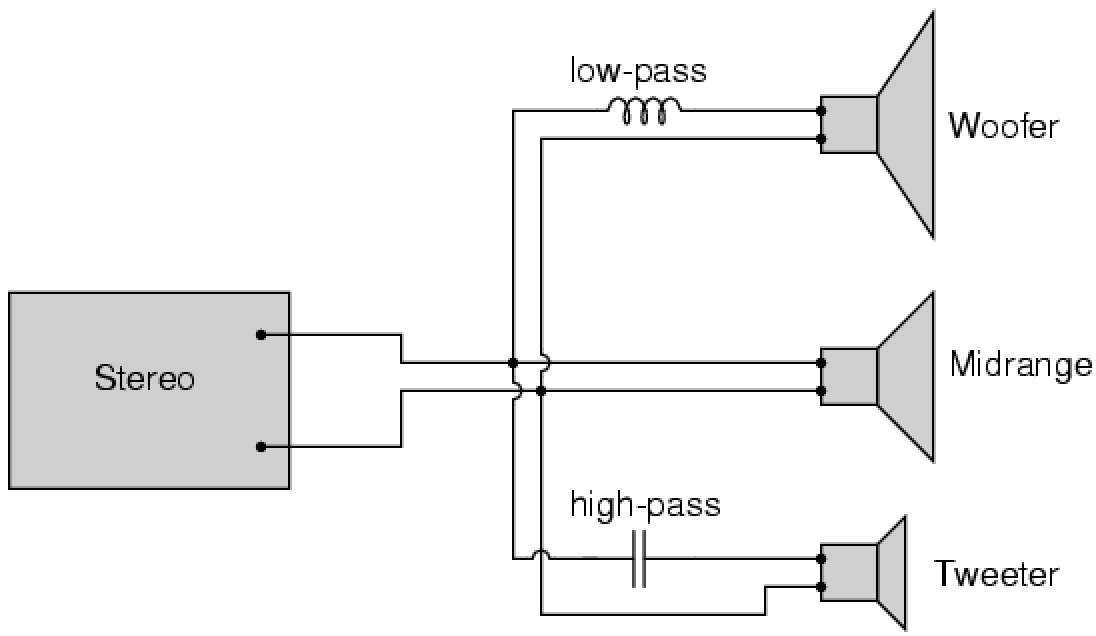


Figure 6.3 When an input signal $v_{in}(t)$ is applied to the input port of a filter, some components are passed to the output port while others are not, depending on their frequencies. Thus, $v_{out}(t)$ contains some of the components of $v_{in}(t)$ but not others. Usually, the amplitudes and phases of the components are altered in passing through the filter.

Filters

Filters process the sinusoid components of an input signal differently depending of the frequency of each component. Often, the goal of the filter is to retain the components in certain frequency ranges and to reject components in other ranges.

Filters



Transfer Functions

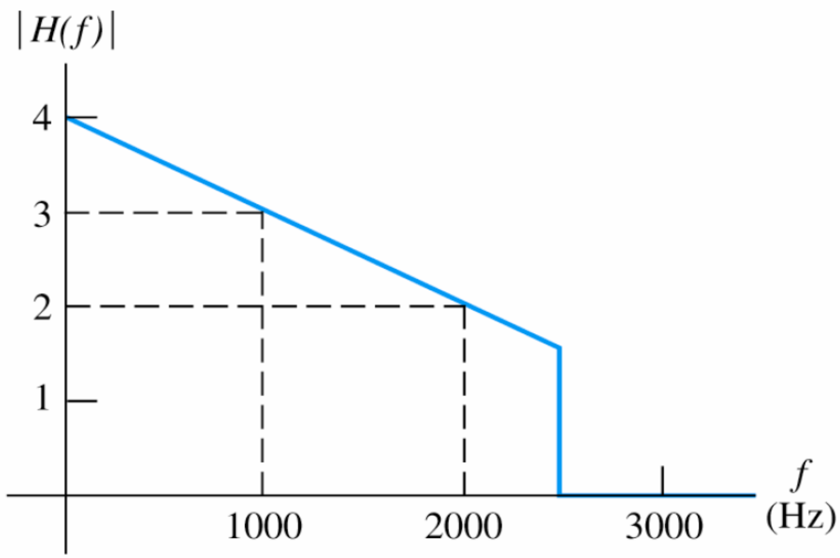
The **transfer function** $H(f)$ of the two-port filter is defined to be the ratio of the phasor output voltage to the phasor input voltage as a function of frequency:

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}$$

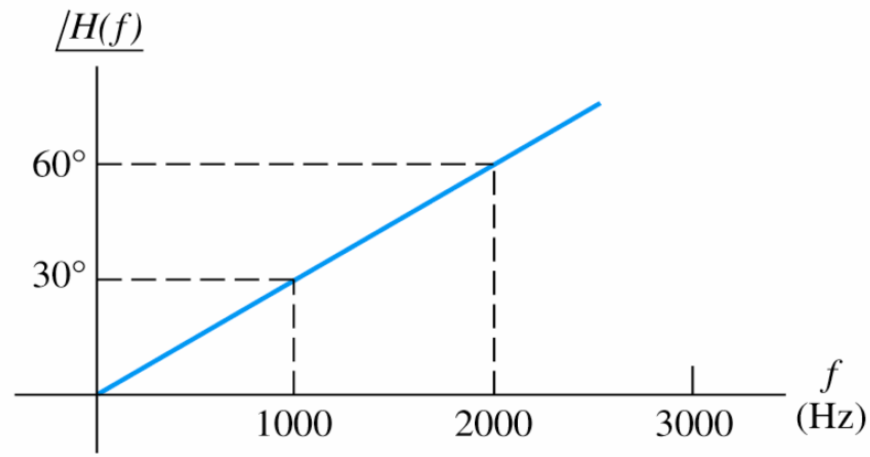
Transfer Functions

The magnitude of the transfer function shows how the amplitude of each frequency component is affected by the filter. Similarly, the phase of the transfer function shows how the phase of each frequency component is affected by the filter.

Transfer Functions

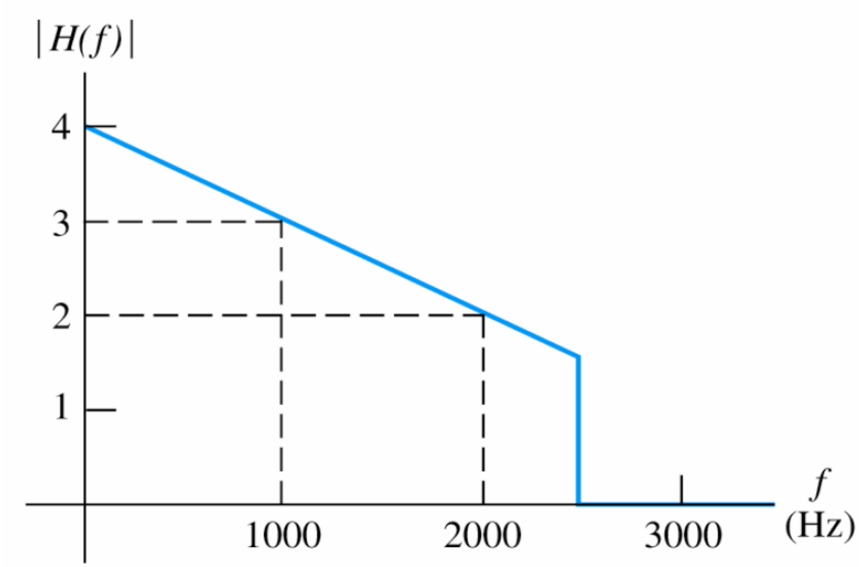


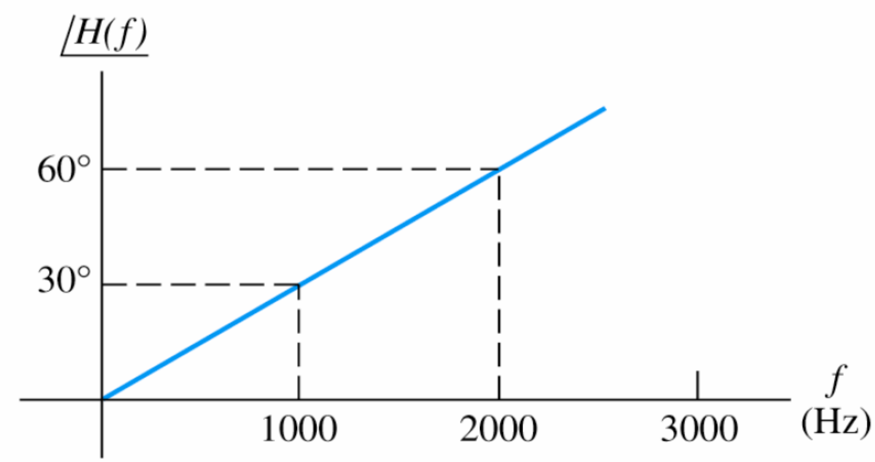
Magnitude



Phase

Example 6.1





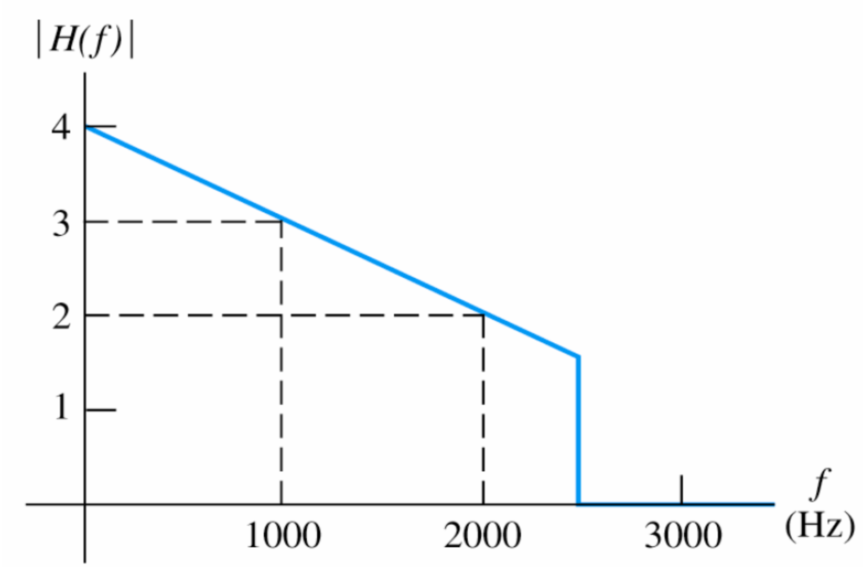
$$v_{in}(t) = 2\cos(2000\pi t + 40^\circ) \rightarrow f = \frac{2000\pi}{2\pi} = 1000\ Hz$$

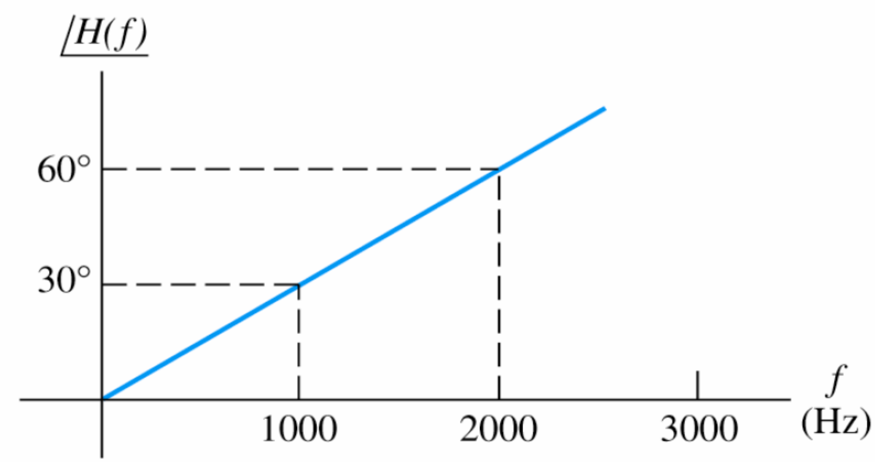
$$\mathbf{V}_{in} = 2\angle 40^\circ$$

$$H(1000) = 3\angle 30^\circ = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} \rightarrow \mathbf{V}_{out} = (3\angle 30^\circ)(\mathbf{V}_{in}) = (3\angle 30^\circ)(2\angle 40^\circ) = 6\angle 70^\circ$$

$$v_{out} = 6\cos(2000\pi t + 70^\circ)$$

Exercise 6.1





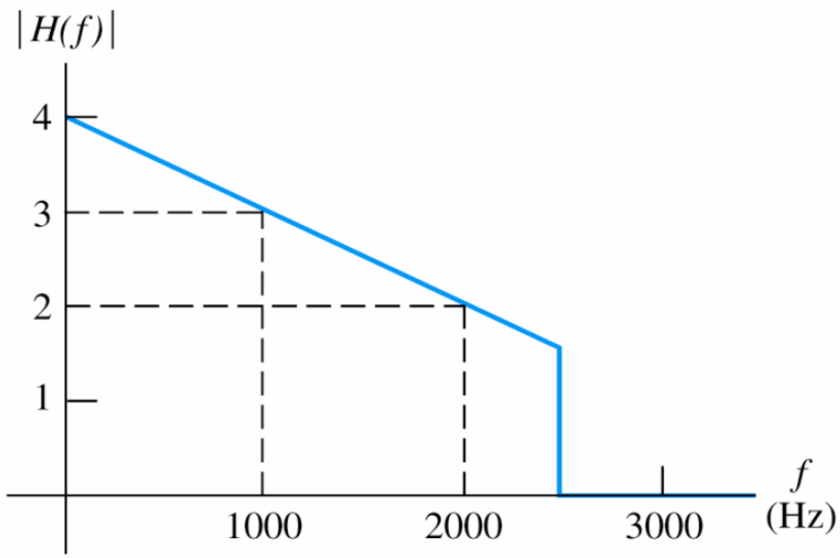
$$v_{in}(t) = 2\cos(4000\pi t) \rightarrow f = \frac{4000\pi}{2\pi} = 2000\ Hz$$

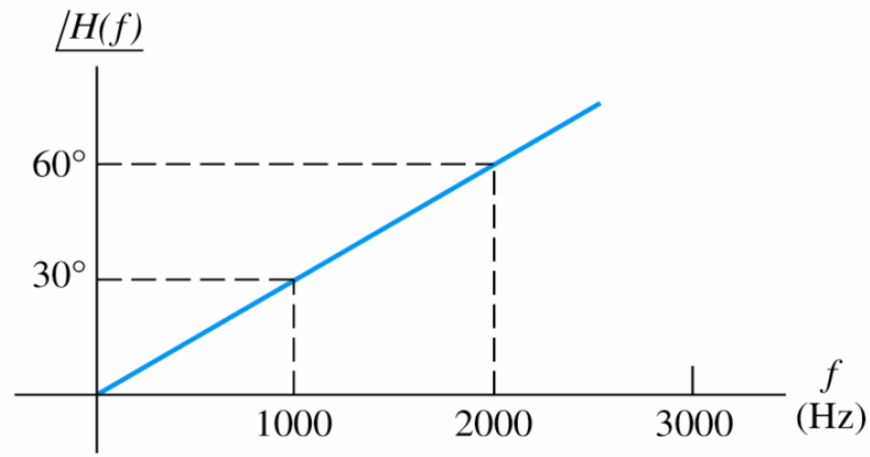
$$\mathbf{V}_{in} = 2\angle 0^\circ$$

$$H(2000) = 2\angle 60^\circ = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} \rightarrow \mathbf{V}_{out} = (2\angle 60^\circ)(\mathbf{V}_{in}) = (2\angle 60^\circ)(2\angle 0^\circ) = 4\angle 60^\circ$$

$$v_{out} = 4\cos(4000\pi t + 60^\circ)$$

Exercise 6.1





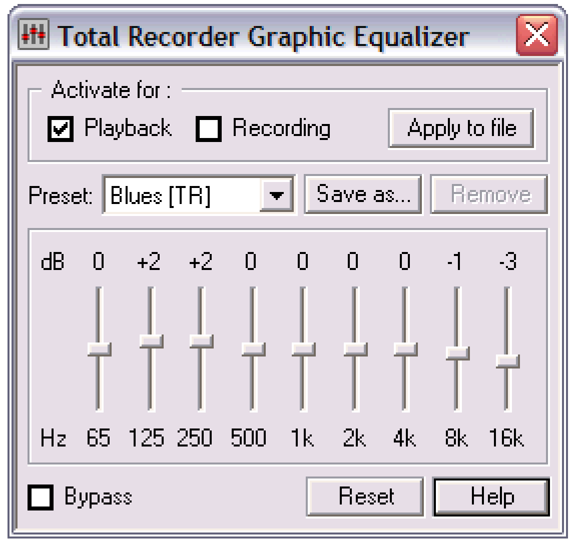
$$v_{in}(t) = 1\cos(6000\pi t - 20^\circ) \rightarrow f = \frac{6000\pi}{2\pi} = 3000\ Hz$$

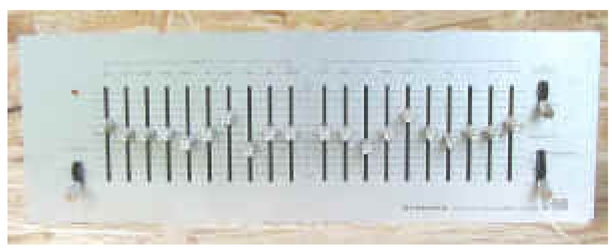
$$\mathbf{V}_{in} = 1\angle 20^\circ$$

$$H(3000) = 0\angle 90^\circ = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} \rightarrow \mathbf{V}_{out} = (0\angle 90^\circ)(\mathbf{V}_{in}) = (0\angle 90^\circ)(1\angle 20^\circ) = 0\angle 110^\circ$$

$$v_{out} = 0\cos(6000\pi t + 110^\circ) = 0$$

Graphic Equalizer





Filter with an adjustable transfer function

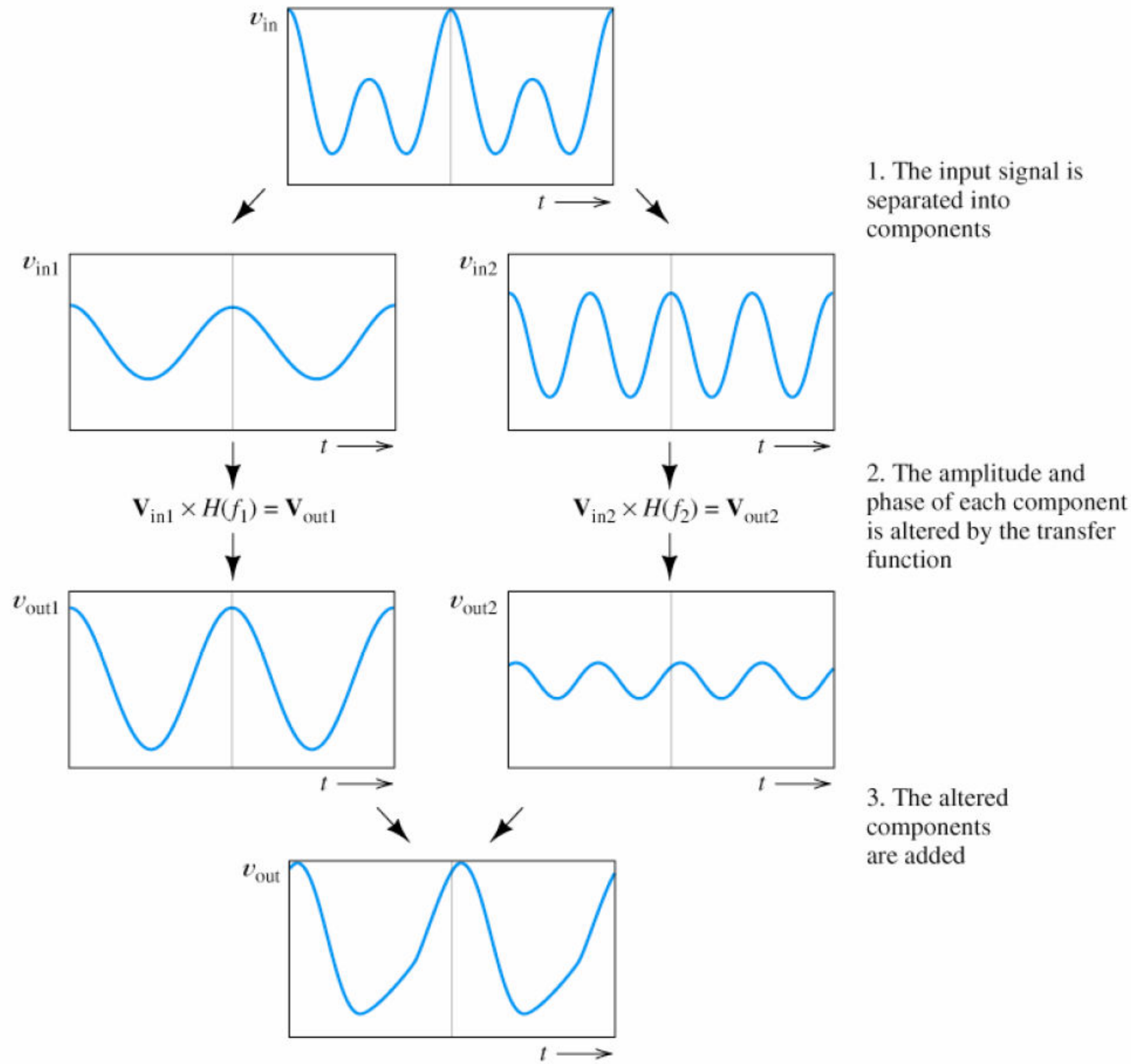
Determining the output of a filter for an input with multiple components:

1. Determine the frequency and phasor representation for each input component.

2. Determine the (complex) value of the transfer function for each component.

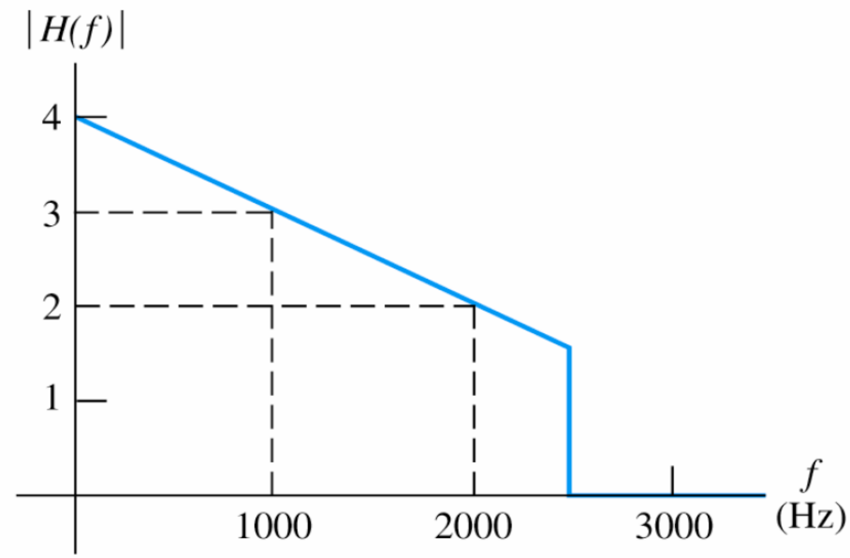
3. Obtain the phasor for each output component by multiplying the phasor for each input component by the corresponding transfer-function value.

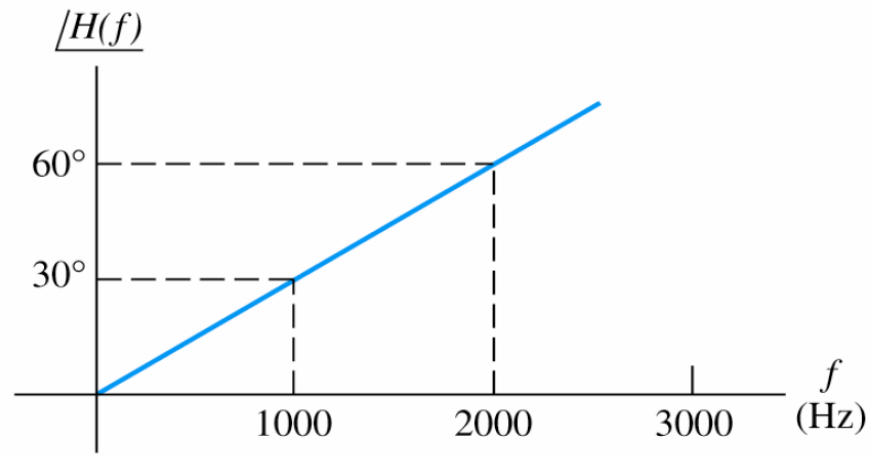
4. Convert the phasors for the output components into time functions of various frequencies. Add these time functions to produce the output.



v_{in}
$t \longrightarrow$
1. The input signal is separated into components
v_{in1}
v_{in2}
$t \longrightarrow$
$t \longrightarrow$
2. The amplitude and phase of each component is altered by the transfer function
$\mathbf{V}_{in1} \times H(f_1) = \mathbf{V}_{out1}$
$\mathbf{V}_{in2} \times H(f_2) = \mathbf{V}_{out2}$
v_{out1}
v_{out2}
$t \longrightarrow$
$t \longrightarrow$
3. The altered components are added
v_{out}
$t \longrightarrow$

Example 6.2





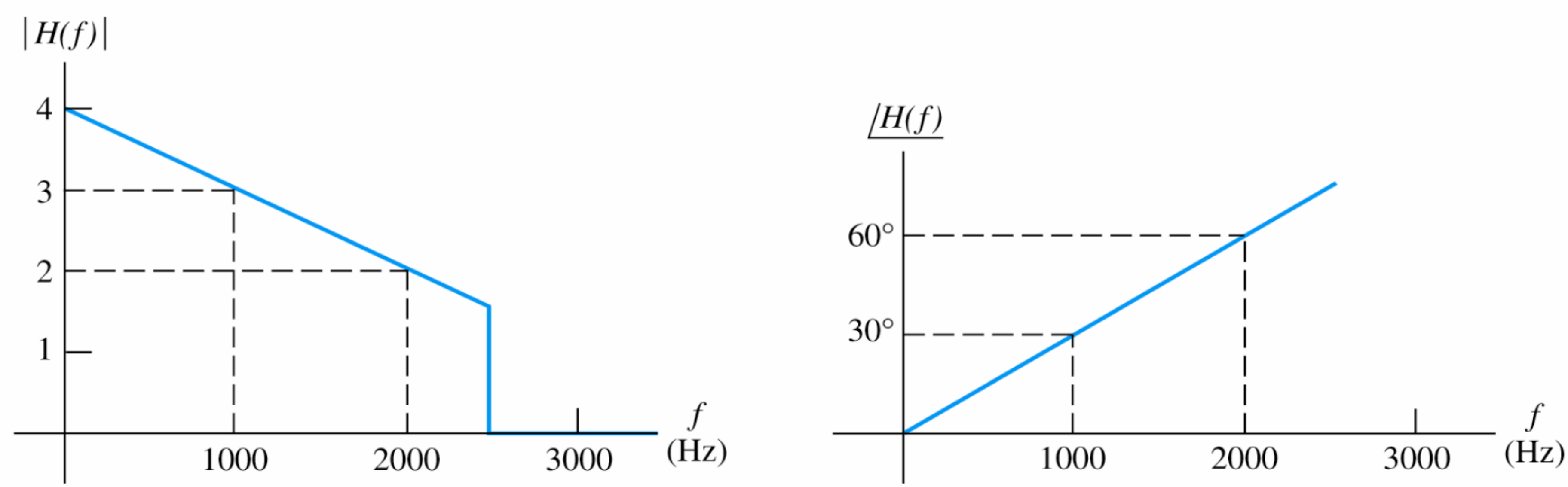
$$v_{in}(t) = 3 + 2\cos(2000\pi t) + \cos(4000\pi t - 70^\circ)$$

$$v_{in1}(t) = 3 \qquad V_{in_1} = 3\angle 0^\circ$$

$$v_{in2}(t) = 2\cos(2000\pi t) \qquad V_{in_2} = 2\angle 0^\circ$$

$$v_{in3}(t) = \cos(4000\pi t - 70^\circ) \qquad V_{in_3} = 1\angle -70^\circ$$

Example 6.2



$H(0) = 4$

$H(1000) = 3\angle 30^\circ$

$H(2000) = 2\angle 60^\circ$

$V_{out_1} = H(0)V_{in_1} = (4\angle 0^\circ)(3\angle 0^\circ) = 12\angle 0^\circ$

$V_{out_2} = H(1000)V_{in_2} = (3\angle 30^\circ)(2\angle 0^\circ) = 6\angle 30^\circ$

$V_{out_3} = H(1000)V_{in_3} = (2\angle 60^\circ)(1\angle -70^\circ) = 2\angle -10^\circ$

$v_{out_1} = 12$

$v_{out_2} = 6\cos(2000\pi t + 30^\circ)$

$v_{out_3} = 2\cos(4000\pi t - 10^\circ)$

$v_{out} = v_{out_1} + v_{out_2} + v_{out_3} = 12 + 6\cos(2000\pi t + 30^\circ) + 2\cos(4000\pi t - 10^\circ)$

Linear circuits behave as if they:

1. Separate the input signal into components having various frequencies.

2. Alter the amplitude and phase of each component depending on its frequency.

3. Add the altered components to produce the output signal.

Determination of the Transfer Function

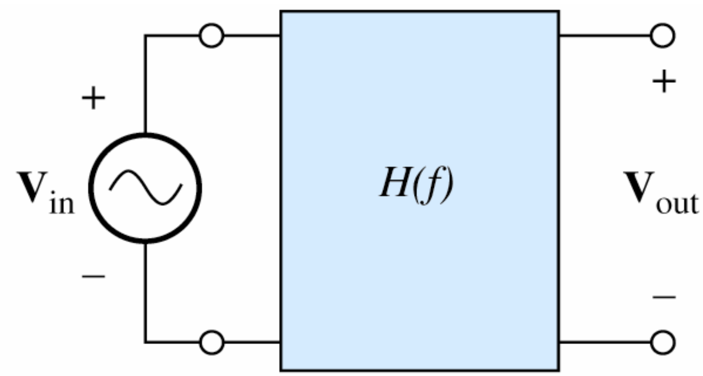
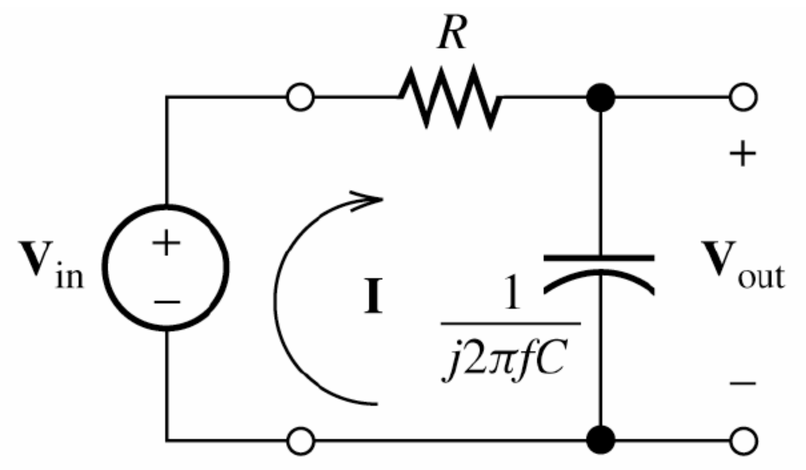


Figure 6.6 To measure the transfer function, we apply a sinusoidal input signal, measure the amplitudes and phases of input and output in steady state, and then divide the phasor output by the phasor input. The procedure is repeated for each frequency of interest.

First-Order Low Pass Filter



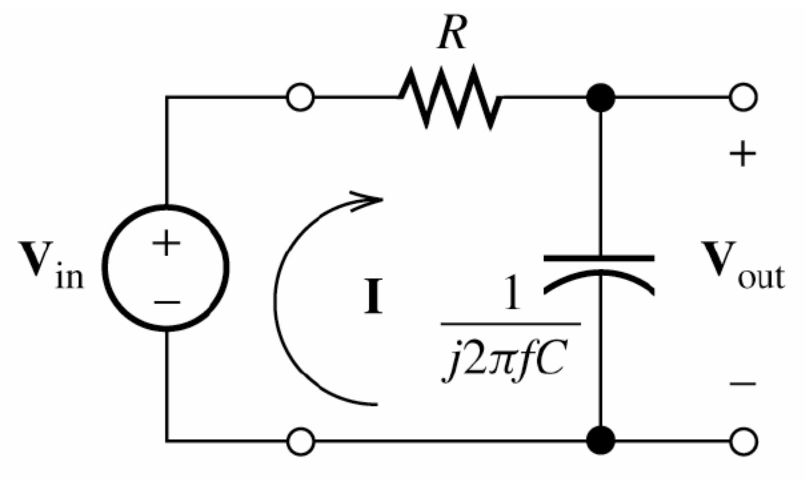
$$\mathbf{I} = \frac{\mathbf{V}_{in}}{R + \frac{1}{j2\pi fC}}$$

$$\mathbf{V}_{out} = \frac{1}{j2\pi fC}\mathbf{I} = \left(\frac{1}{j2\pi fC}\right)\left(\frac{\mathbf{V}_{in}}{R + \frac{1}{j2\pi fC}}\right)$$

$$H(f) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{1}{1 + j2\pi fRC} = \frac{1}{1 + j(f / f_B)} \qquad f_B = \frac{1}{2\pi RC}$$

Half power frequency

First-Order Low Pass Filter



$$H(f) = \frac{1}{1 + j(f/f_B)} = \frac{1\angle 0^\circ}{\sqrt{1 + (f/f_B)^2}\angle \arctan\left(\frac{f}{f_B}\right)}$$

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \qquad \angle H(f) = -\arctan\left(\frac{f}{f_B}\right)$$

First-Order Low Pass Filter

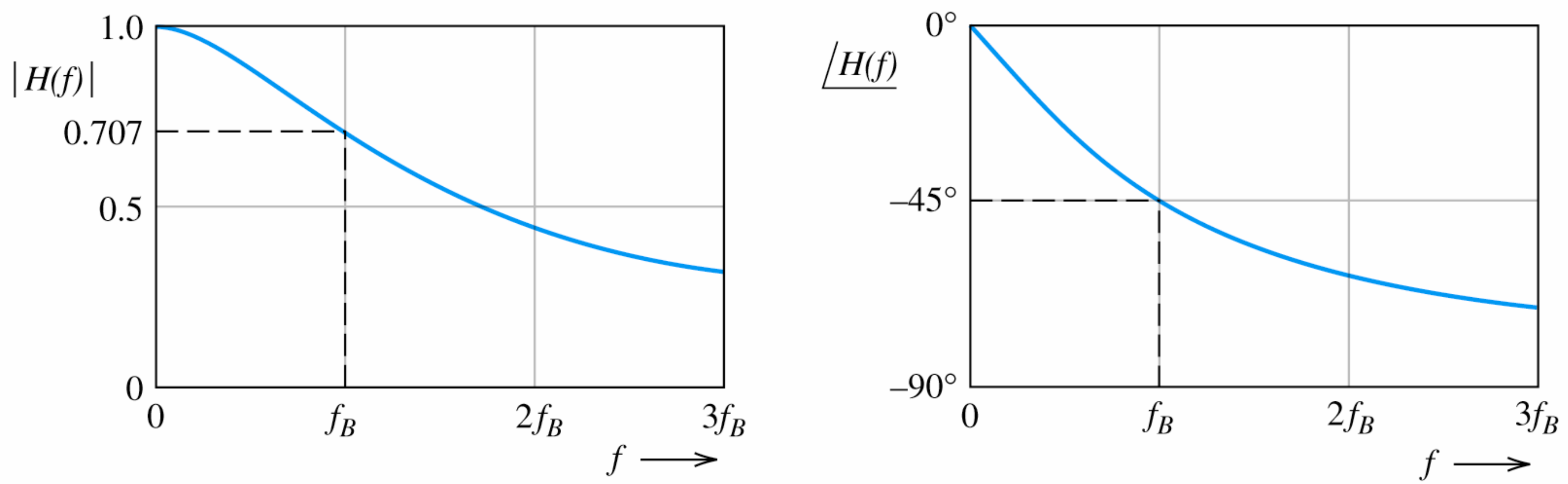
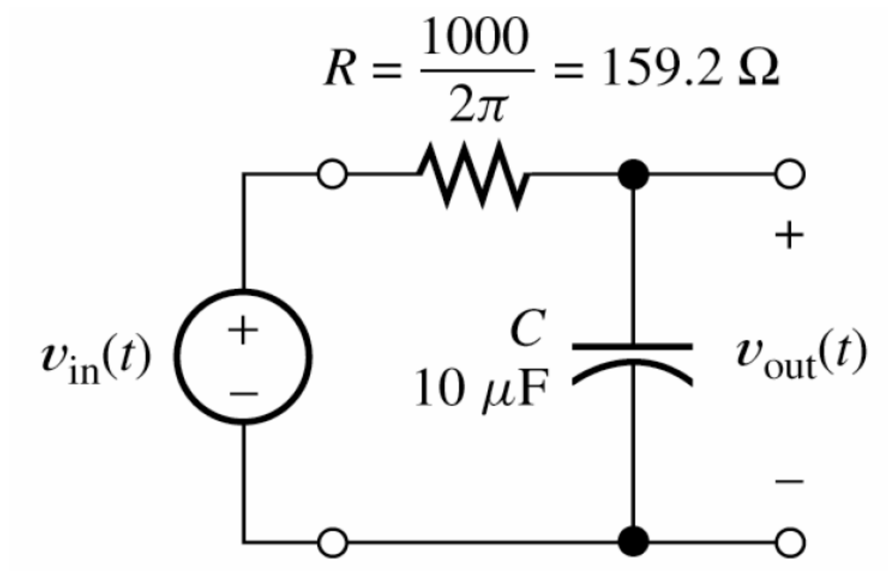


Figure 6.8 Magnitude and phase of the first-order lowpass transfer function versus frequency.

For low frequency signals the magnitude of the transfer function is unity and the phase is 0°. Low frequency signals are passed while high frequency signals are attenuated and phase shifted.

Calculation of RC Low-Pass Output



$$v_{in} = 5\cos(20\pi t) + 5\cos(200\pi t) + 5\cos(2000\pi t)$$

$$f_B = \frac{1}{2\pi RC} = \frac{1}{2\pi\left(\frac{1000}{2\pi}\right)(10x10^{-6})} = 100Hz$$

Calculation of RC Low-Pass Output

$$\left|H(f)\right| = \frac{1}{\sqrt{1+\left(f/100\right)^2}} \qquad \angle H(f) = -\arctan\left(\frac{f}{100}\right)$$

$$v_{in_1} = 5\cos(20\pi t) \qquad \mathbf{V}_{in_1} = 5\angle 0^\circ \qquad H(10) = \frac{1}{1+j(10/100)} = 0.9950\angle -5.71^\circ$$

$$v_{in_2} = 5\cos(200\pi t) \qquad \mathbf{V}_{in_2} = 5\angle 0^\circ \qquad H(100) = \frac{1}{1+j(100/100)} = 0.7071\angle -45^\circ$$

$$v_{in_3} = 5\cos(2000\pi t) \qquad \mathbf{V}_{in_3} = 5\angle 0^\circ \qquad H(1000) = \frac{1}{1+j(1000/100)} = 0.0995\angle -84.29^\circ$$

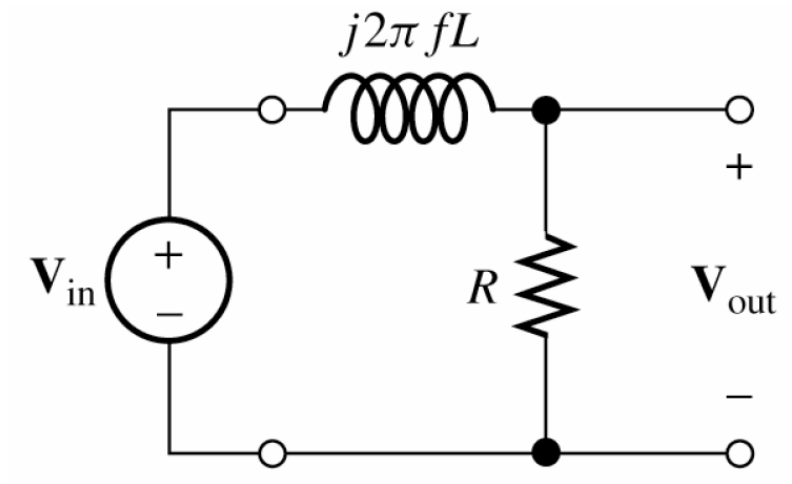
$$\mathbf{V}_{out_1} = H(10)\mathbf{V}_{in_1} = (0.9950\angle -5.71^\circ)(5\angle 0^\circ) = 4.975\angle -5.71^\circ \qquad v_{out_1} = 4.975\cos(20\pi t - 5.71^\circ)$$

$$\mathbf{V}_{out_2} = H(100)\mathbf{V}_{in_2} = (0.7071\angle -45^\circ)(5\angle 0^\circ) = 3.535\angle -45^\circ \qquad v_{out_2} = 3.535\cos(200\pi t - 45^\circ)$$

$$\mathbf{V}_{out_3} = H(1000)\mathbf{V}_{in_3} = (0.0995\angle -84.29^\circ)(5\angle 0^\circ) = 0.4975\angle -84.29^\circ \qquad v_{out_3} = 0.4975\cos(2000\pi t - 84.29^\circ)$$

$$v_{out}(t) = v_{out_1}(t) + v_{out_2}(t) + v_{out_3}(t) = 4.975\cos(20\pi t - 5.71^\circ) + 3.535\cos(200\pi t - 45^\circ) + 0.4975\cos(2000\pi t - 84.29^\circ)$$

Exercise 6.4

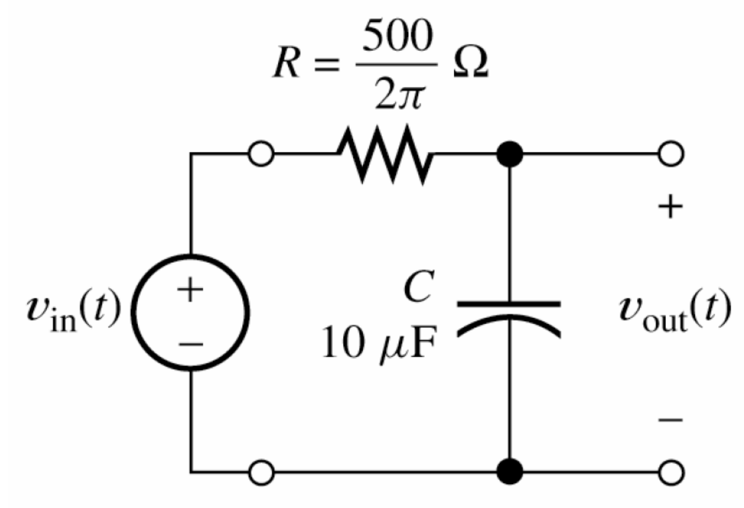


$$\mathbf{I} = \frac{\mathbf{V}_{in}}{R + (2\pi f L)\,j}$$

$$\mathbf{V}_{out} = \mathbf{I}R = \left(\frac{\mathbf{V}_{in}}{R + (2\pi f L)\,j}\right)R$$

$$H(f) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{R}{R + j2\pi f L} = \frac{1}{1 + j\dfrac{2\pi L f}{R}} = \frac{1}{1 + j(f / f_B)} \qquad f_B = \frac{R}{2\pi L}$$

Exercise 6.5



$$v_{in}(t) = 10\cos(40\pi t) + 5\cos(1000\pi t) + 5\cos(2\pi 10^4 t)$$

$$f_B = \frac{1}{2\pi RC} = \frac{1}{2\pi \frac{500}{2\pi} 10x10^{-6}} = \frac{1}{5x10^{-3}} = 0.2x10^2 = 200Hz$$

$$H(f) = \frac{1}{1 + (f / f_B)j} = \frac{1}{1 + (f/200)j}$$

$v_1(t) = 10\cos(40\pi t)$ $\quad \mathbf{V}_1 = 10\angle 0^\circ$ $\quad H(20) = \frac{1}{1+0.1j} = 0.995\angle -5.711^\circ$ $\quad \mathbf{V}_{1_{out}} = (10\angle 0^\circ)(0.995\angle -5.711^\circ) = 9.95\angle -5.711^\circ$

$v_2(t) = 5\cos(1000\pi t)$ $\quad \mathbf{V}_2 = 5\angle 0^\circ$ $\quad H(500) = \frac{1}{1+2.5j} = 0.3714\angle -68.2^\circ$ $\quad \mathbf{V}_{2_{out}} = (5\angle 0^\circ)(0.3714\angle -68.2^\circ) = 1.857\angle -68.2^\circ$

$v_3(t) = 5\cos(2\pi 10^4 t)$ $\quad \mathbf{V}_3 = 5\angle 0^\circ$ $\quad H(10{,}000) = \frac{1}{1+50j} = 20x10^{-3}\angle -88.85^\circ$ $\quad \mathbf{V}_{3_{out}} = (5\angle 0^\circ)(20x10^{-3}\angle -88.85^\circ) = 0.1\angle -88.85^\circ$

$$v_{out} = 9.95\cos(40\pi t - 5.71^\circ) + 1.85\cos(1000\pi t - 68.2^\circ) + 0.10\cos(2\pi 10^4 t - 88.9^\circ)$$

Decibels, the Cascade Connection, and Logarithmic Frequency Scales

$$\left|H(f)\right|_{\text{dB}} = 20\log\left|H(f)\right|$$

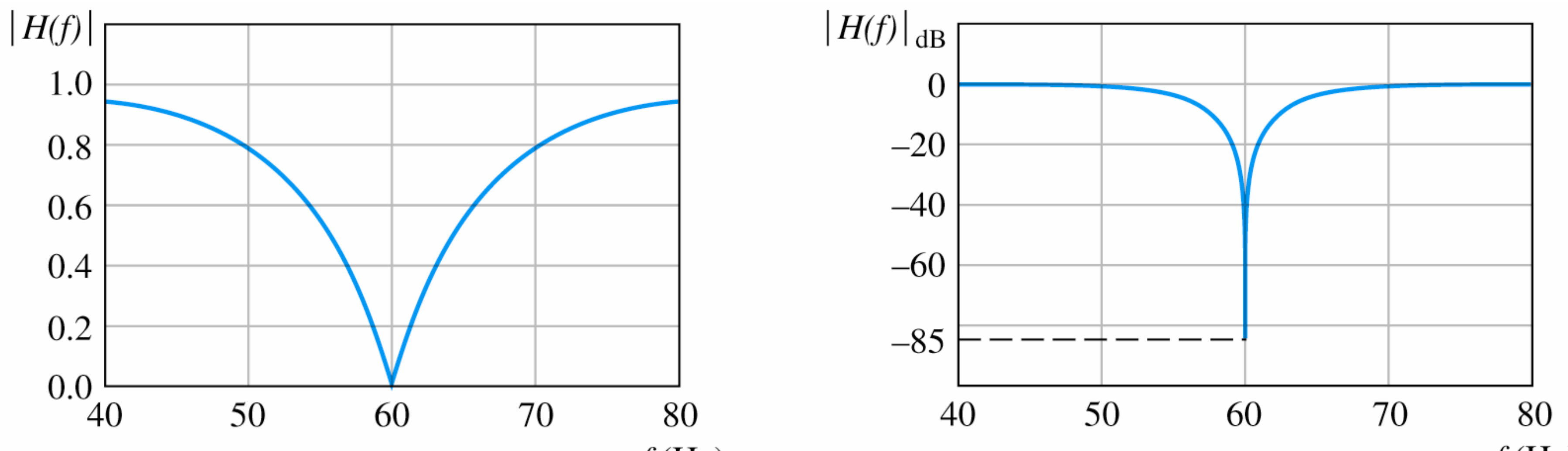
$$\text{dB} \rightarrow \text{decibels}$$

Table 6.2. Transfer-Function Magnitudes and Their Decibel Equivalents

$\|H(f)\|$	$\|H(f)\|_{dB}$
100	40
10	20
2	6
$\sqrt{2}$	3
1	0
$1/\sqrt{2}$	−3
1/2	−6
0.1	−20
0.01	−40

Numbers greater than 1 are positive

Numbers smaller than 1 are negative



(a) Linear scale

(b) Decibel scale

Figure 6.12 Transfer-function magnitude of a notch filter used to reduce hum in audio signals.

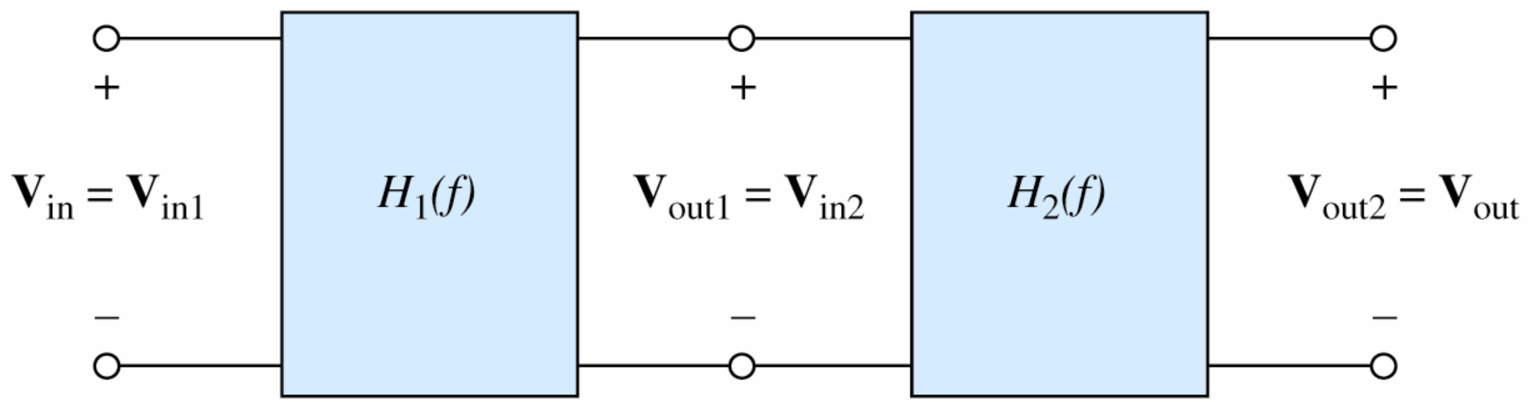


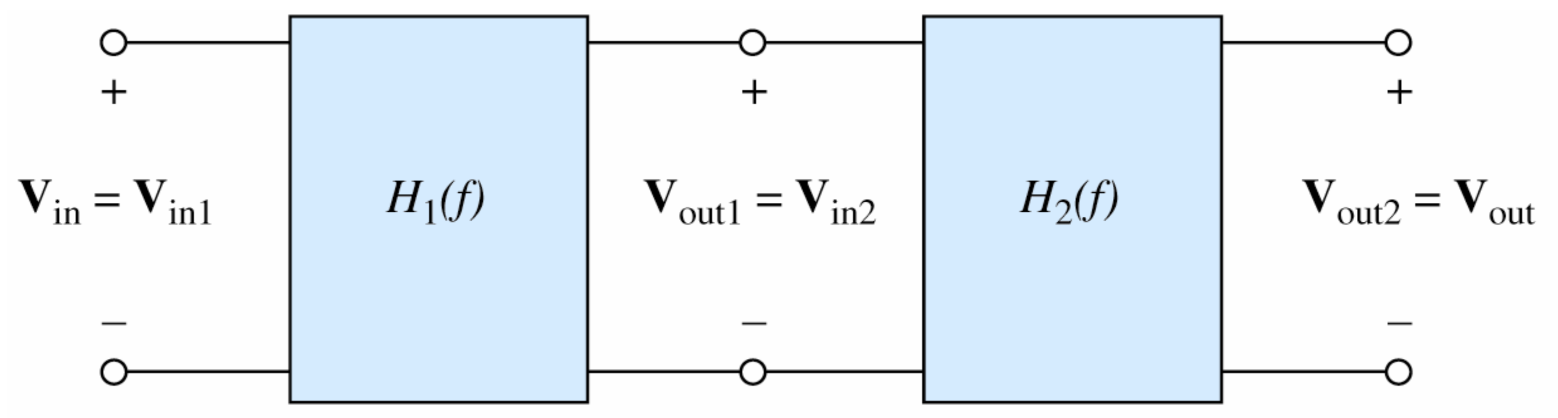
Figure 6.13 Cascade connection of two two-port circuits.

$$H(f) = \frac{V_{out_2}}{V_{in_1}} = \frac{V_{out_1}}{V_{in_1}} \frac{V_{out_2}}{V_{out_1}} = \frac{V_{out_1}}{V_{in_1}} \frac{V_{out_2}}{V_{in_2}} = H_1(f)H_2(f)$$

Cascaded Two-Port Networks

$$H(f) = H_1(f) \times H_2(f)$$

$$|H(f)|_{dB} = |H_1(f)|_{dB} + |H_2(f)|_{dB}$$



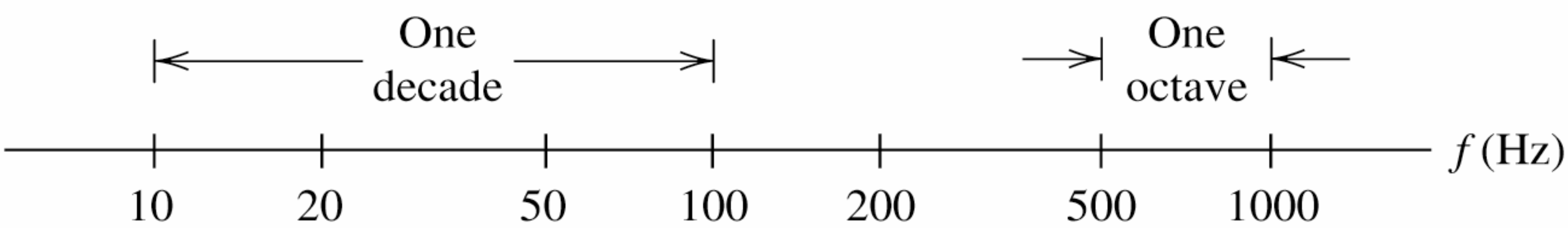


Figure 6.14 Logarithmic frequency scale.

One decade: $$\frac{f_h}{f_l} = 10$$

One octave: $$\frac{f_h}{f_l} = 2$$

Logarithmic Frequency Scales

On a logarithmic scale, the variable is multiplied by a given factor for equal increments of length along the axis.

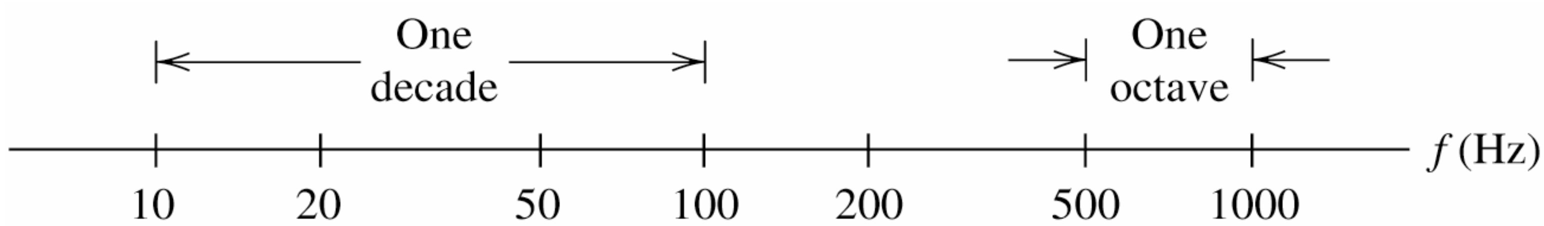


Figure 6.14 Logarithmic frequency scale.

A **decade** is a range of frequencies for which the ratio of the highest frequency to the lowest is 10.

$$\text{Number of decades} = \log\left(\frac{f_2}{f_1}\right)$$

An **octave** is a two-to-one change in frequency.

$$\text{Number of octaves} = \log_2\left(\frac{f_2}{f_1}\right) = \left(\frac{\log(f_2/f_1)}{\log(2)}\right)$$

Exercise 6.6

Convert the magnitude of the transfer function H(f) into dB's:

$$|H(f)| = 50$$

$$|H(f)|_{dB} = 20\log(50) = 34dB$$

Exercise 6.7

$$(a)\ \left|H(f)\right|_{dB} = 15dB = 20\log\left|H(f)\right| \rightarrow \left|H(f)\right| = 10^{\frac{15}{20}} = 5.623$$

$$(b)\ \left|H(f)\right|_{dB} = 30dB = 20\log\left|H(f)\right| \rightarrow \left|H(f)\right| = 10^{\frac{30}{20}} = 31.62$$

Exercise 6.8

(a) What frequency is two octaves higher than 1000 Hz?

$$2 = \log_2\left(\frac{f_2}{1000}\right) = \frac{\log\left(\frac{f_2}{1000}\right)}{\log(2)}$$

$$\log\left(\frac{f_2}{1000}\right) = 2\log(2)$$

$$\frac{f_2}{1000} = 10^{2\log(2)}$$

$$f_2 = 1000x10^{2\log(2)} = 4{,}000$$

Exercise 6.8

(b) What frequency is three octaves lower than 1000 Hz?

$$3 = \log_2\left(\frac{1000}{f_1}\right) = \frac{\log\left(\frac{1000}{f_1}\right)}{\log(2)}$$

$$\log\left(\frac{1000}{f_1}\right) = 3\log(2)$$

$$\frac{1000}{f_1} = 10^{3\log(2)}$$

$$f_1 = \frac{1000}{10^{3\log(2)}} = 125$$

Exercise 6.8

(c) What frequency is one decade lower than 1000 Hz?

$$1 = \log\left(\frac{1000}{f}\right) \rightarrow \frac{1000}{f} = 10$$

$$f = 100Hz$$

Exercise 6.9

(a) What frequency is halfway between 100 and 1000 Hz on a logarithmic frequency scale?

$$\log(100) = 2$$

$$\log(1000) = 3$$

$$average = \frac{3+2}{2} = 2.5$$

$$\log(f) = 2.5 \rightarrow f = 10^{2.5} = 316.2Hz$$

Exercise 6.9

(b) What frequency is halfway between 100 and 1000 Hz on a linear frequency scale?

$$f_1 = 100Hz$$

$$f_2 = 1000Hz$$

$$f = \frac{1000Hz + 100Hz}{2} = 550Hz$$